PERFCTOID AND PRISMATIC METHODS

1. Getting acquainted

Exercise 1.1. Let A be a p-complete ring. Show that if $\mathfrak{m}$ is a maximal ideal in A then $p \in \mathfrak{m}$.

Exercise 1.2. Let $M \to N$ be a map p-complete modules if the reduction modulo p is injective/surjetive/bijective, what can you say about the map? And if in addition M and N are p-torsion free?

Exercise 1.3. Let M be a p-complete module, p-torsion free. If M/pM is free, what can you say about M?

Exercise 1.4. Let $R \to S$ and $R \to T$ map of perfect rings in characteristic p. Show that

$$S \otimes_R^L T \to S \otimes_R T$$

is an isomorphism.

Exercise 1.5. Show that a Noetherian perfect ring is a finite product of perfect fields.

Exercise 1.6. Let R be a normal domain such that it's fraction field is algebraically closed. Show that the Zariski and étale site of R are the same.

Now let X be an integral normal scheme with algebraically closed function field. What about the étale cohomology of X?

Exercise 1.7. Let R be a normal domain and $R \to S$ finite. Suppose they are $\mathbb{Q}$-algebras. The show that there is a module theoretic section.

Exercise 1.8. Let A be a δ-ring such that ϕ is injective. Show that then A is p-torsion free. Deduce that the Witt vectors of a perfect ring are p-torsion free.

Exercise 1.9. Let A be a δ-ring. Show that for any $a, b \in A$ we have $\delta(a^p + pb) \equiv b^p$ modulo pA. Hint: by induction on n show that $\delta(x^n) \equiv nx^{(n-1)p}\delta(x)$ modulo $p\delta(x)$. Deduce that (also by induction) $\delta(x^{p^n}) \in p^n A$.

Exercise 1.10. Consider $\mathbb{Z}\{x\} = \mathbb{Z}[x_0, x_1, \ldots, x_n, \ldots]$, with $x = x_0$, the free δ-ring on one element.

(1) Show by induction that they are $y_n \in \mathbb{Z}\{x\}$ such that $\delta^{n-1}(px_0) = y_{n-1}^p + py_n$ for $n \geq 1$ and $y_0 = 0$. What is y_1 in particular? Show also that $y_{n+1} = x_n + q_n(x_0, \ldots, x_{n-1})$ for some integral polynomial q_n which vanishes when evaluated at the origin. Deduce that for any ring R and $k \geq 1$

$$V_R^k(W(R)) = \{(a_n)_{n\in\mathbb{N}} \in W(R) \mid a_i = 0 \quad i \leq k-1\}$$

(2) Now consider

$$V^* \colon \mathbb{Z}\{x\} \to \mathbb{Z}\{x\} \quad x_n \mapsto y_n.$$

Show that $V^*\phi = \cdot p$, where $\cdot p$ is the unique δ-ring map sending $x \mapsto px$.

(3) If R is a perfect ring in characteristic p, show that the image of V_R is equal to the ideal $pW(R)$. Deduce that $W(R)/pW(R) = R$.
(4) Deduce also that $W(R)$ is p-complete when R is a perfect ring in characteristic p.