PERFCTOID AND PRISMATIC METHODS

2. Perfectoid rings

Exercise 2.1. Let A be a p-complete delta ring which is perfect modulo p. Let $d \in A$ be distinguished. Show that A is d-torsion free.

Exercise 2.2. Recall the definition of perfectoid ring.

A ring R is called *perfectoid* if

(1) It is p-complete,
(2) There is $\varpi \in R$ with $\varpi^p = pu$ for some unit $u \in R$,
(3) The map $\theta\colon A_{\mathrm{inf}}(R) \to R$ is surjective with principal kernel.

Show that (2) implies that $\ker(\theta)$ is distinguished. Deduce that the following definition is equivalent.

(1) It is p-complete,
(2) The map $\theta\colon A_{\mathrm{inf}}(R) \to R$ is surjective with principal kernel generated by a distinguished element.

Exercise 2.3. Let R be a ring and I be an ideal. Show that $(r + I^n)_{n\in\mathbb{N}}$ form a basis for a topology that we call the I-adic topology. Cauchy sequences are sequences such that $a_m - a_n \in I^N$ for some $m, n \geq M$, an M that exists for any prescribed N. The property of being I-complete is defined with respect to these Cauchy sequences. Show that a ring is I-complete in the above sense if and only if the natural map

$$R \to \varprojlim_{n\in\mathbb{N}} R/I^n$$

is an isomorphism.

Exercise 2.4. (Glimpse at deformation theory.) Let $A \to B$ be a ring map. Then there exists $L_{B/A} \in \mathcal{D}(B)_{\geq 0}$ such that

$$\pi_0(L_{B/A}) = \Omega_{B/A}$$

and such that $L_{B/A}$ controls (higher) deformations of B as an A-algebra.

If $L_{B|A} = 0$ then a consequence is that $A \to B$ is *formally étale.* This means that for any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & R \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & R/I \end{tikzcd}$$

with $I^2 = 0$, then there exist a unique A-algebra lift as in the diagram.

Such considerations are really useful in view of the following exercise. Let S be a perfect ring in characteristic p. Show that $\Omega_{S/\mathbb{F}_p} = 0$. (This is used to prove that the same holds for the derived analogue: $L_{S/\mathbb{F}_p} = 0$. Therefore any perfect ring is formally étale as an $\mathbb{F}_p$-algebra. This allows for strong lifting properties.)

Exercise 2.5. Let R be a ring and $\varpi \in R$ such that $\varpi^p = pu$ for some unit $u \in R$. Show that

$$(-)^p \colon R/\varpi R \to R/pR$$

is injective if and only if $R \to R[1/p]$ is *p-integrally closed*, meaning that if $x \in R[1/p]$ with $x^p \in R$ then $x \in R$.

Exercise 2.6. Find by hand a ϖ in $R = \mathbb{Z}_p^{\mathrm{cycl}}$ that fits the definition of perfectoid rings such that

$$(-)^p \colon R/\varpi R \to R/pR$$

is bijective.

Exercise 2.7. Show that the p-completion of an absolutely integrally closed domain is perfectoid.

Exercise 2.8. Give examples of perfectoid rings, and show that they are perfectoid. Try to give at least one example of perfectoid ring with p-torsion, but not in characteristic p.

Exercise 2.9. Is the p-completion of $\mathcal{O}_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}$ perfectoid?

Exercise 2.10. Is there an initial perfectoid ring?