Perfectoid and prismatic methods

Context:

$$X^n + Y^n = Z^n \quad n \in \mathbb{N}$$

- Solutions over $\mathbb{C} \rightsquigarrow$ class. alg. geom
- Solution over $\mathbb{Z}$ (finite ext $\mathbb{Z}$/ or $\mathbb{Q}$)
  $\rightsquigarrow$ arithmetic geometry

Pick a prime

$p \to$ What "locally happens around p"

Def Let M be a module.

It's p-completion

$$M \xrightarrow{(*)} M^{\wedge p} := \varprojlim_n M/p^n M$$

$$M/p^nM \to M/p^{n-1}M$$

M is p-complete if $(*)$ is an iso.

Rem If M is a derived module (a complex) above def makes sense $\to$ "derived completion".

1) Goals / examples of applications of said methods.

a) **Def** (Splinter) R is a splinter if $\forall R \to S$ finite
$\exists \sigma : S \to R$ R-linear section.

In char 0: normal domain Noetherian
splinter $\Leftrightarrow$ normal

In char p: Hochster proved

[Regular, Noeth $\Rightarrow$ Splinter]

Thm (André) R reg. Noeth
$$\Downarrow$$
R splinter

"Perfectoid methods" were heavily used

Main tool in our seminar: "RH for perfectoid rings".

b) Almost Kodaira vanishing

Thm $K/\mathbb{Q}_p$ a perfectoid field

$\mathfrak{X}/\mathcal{O}_K$ proj flat rel dim d with ample $L \in \mathrm{Pic}(\mathfrak{X})$

$$R\Gamma(\mathfrak{X}^{pfd}, L^{-1}) \overset{a}{=} 0$$

$\to$ When there is "good hypothesis" on $\mathfrak{X}$ then you can have a more "honest" form of Kodaira Vanishing.

Remark With a gen. of the above you can reprove (KAN) in char 0, it gives a p-adic proof.

And more... (e.g. Bhatt's result "Kodaira vanishing up to finite cover")

2) The tools

$K/\mathbb{Q}_p$ finite extension.

$\mathfrak{X}/\mathcal{O}_K$ a finite "p-adic formal" scheme

$\mathfrak{X}_\eta =: X$ adic generic fibre.

$$\text{RH}: \widehat{\mathcal{D}}(X_{\acute{e}t}, \mathbb{Z}_p)^{oc} \underset{\text{fully-faithful}}{\longrightarrow} \overbrace{\mathrm{Crys}(\overline{\mathbb{A}}_{\mathfrak{X}})^a}^{\text{(almost) perfectoid crystals}} \; (*)$$

RH: Riemman-Hilbert

$\widehat{\mathcal{D}}$: p-complete objects

$\mathbb{Z}_p$: p-completion of $\mathbb{Z} = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$

$$(*) \quad \mathcal{D}_{qc}(\mathfrak{X}^{pfd}) = \mathrm{Crys}(\overline{\mathbb{A}}_{\mathfrak{X}})$$

Goal(s): Understand the terms / properties / proofs / applications.

Closely related "prismatic cohomology".

$\mathfrak{X}/\mathcal{O}_K$

lift of Frobenius

[ϕ-perfection]

$$\begin{tikzcd}
\mathfrak{X}^{dR} \arrow[dr] & & \\
& \mathfrak{X}^{\Delta} \arrow[r] & \mathfrak{X}^{\Delta}_{perf} \arrow[d] \\
\mathfrak{X}^{HT} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathfrak{X}^{pfd} \arrow[dl] \\
& \mathfrak{X} &
\end{tikzcd}$$

When you take $\mathcal{D}_{qc}(-)$, give rise to different cohomology theories.

- $\mathcal{D}_{qc}(\mathfrak{X}) \to$ quasi-coherent cohomology
- $\mathcal{D}_{qc}(\mathfrak{X}^{pfd}) \to$ perfectoid crystals
- $\mathcal{D}_{qc}(\mathfrak{X}^{\Delta}) \to$ prismatic crystals

$$\mathfrak{X}^{\Delta} \xrightarrow{f} \mathbb{Z}_p^{\Delta}$$

$$R\Gamma(f_* \mathcal{O}_{\mathfrak{X}^{\Delta}}) = H_{\Delta}(\mathfrak{X})$$

(absolute prismatic)

- $\mathcal{D}_{qc}(\mathfrak{X}^{dR}) =$ "a-version of $\mathcal{D}$-modules"

$$\mathfrak{X}^{dR} \xrightarrow{\rho_{dR}} \mathfrak{X}^{\Delta}$$

$$\rho_{dR}^* H_{\Delta}(\mathfrak{X}) = H^{dR}_{\mathbb{Z}_p}(\mathfrak{X})$$

- Dates:
  - Scholze (2012) came up with gen. pfd spaces
  - Bhatt-Scholze (2018) prismatic cohomology
  - Bhatt-Lurie (2020) RH functor and applications (Bhatt).

$\mathcal{X}$ is a formal scheme.

$$\begin{array}{c} \mathcal{X} \\ \downarrow (-)_\eta \\ X_\eta \end{array}$$

" taking $\underbrace{\text{adic}}$ generic fibre"

(An adic/rigid space)

→ Keep track of p-adic topology.

A p-complete ring

$$\mathrm{Spa}(A[1/p], A)$$

$\mathcal{O}_X$ $\mathcal{O}_X^+$

quasi-coherent

the topology and the complete aspect.

→ Comparison between Vector bundles on the "usual generic fibre" = "adic vector bundles"

→ Gaga-theorems

3) δ-Rings, Witt vectors, perfectoid rings.

Def (δ-Ring) Let R be a ring. A δ-ring structure on R

$$\delta : R \to R \quad (\text{function})$$

i) $\delta(1) = 0$ ii) $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x^i y^{p-i}$

iii) $\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\,\delta(x)\,\delta(y)$. □

Remark If R is p-torsion free

$$\delta\text{-Structure on } R = \left(\begin{array}{l} \phi : R \to R \text{ s.t.} \\ \phi : R/pR \to R/pR \text{ is} \\ \text{Frobenius } (-)^p \end{array} \right)$$

$$\xrightarrow[\text{a } \phi \text{ lift of Frob}]{\text{Always get}} \phi(x) = x^p + p\,\delta(x) \text{ is a ring morphism.}$$ □

Examples i) $\mathbb{Z}_p = \left(\lim_{\leftarrow n} \mathbb{Z}/p^n\mathbb{Z}\right)$, $\mathbb{Z}$, $\mathbb{Z}_{(p)}$

$\phi = \mathrm{id}$.

ii) $\mathbb{Z}[x]$ $\phi(x) = x^p + p\,\delta(x)$

$\delta(x) \in \mathbb{Z}[x]$

iii) Remark/exercise if $p = 0$ in R then there is no δ-ring struc on R (unless $R = 0$)

Remark R a ring

$$W_2(R) \overset{\text{as a set}}{=} R \times R$$

$$(x_0, x_1) \hat{+} (y_0, y_1) = \left(x_0 + y_0,\ x_1 + y_1 - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x_0^i y_0^{p-i}\right)$$

$$(x_0, x_1) \hat{\cdot} (y_0, y_1) = \left(x_0 \cdot y_0,\ x_0^p y_1 + x_1 y_0^p + p x_1 y_1\right)$$

A δ-ring structure
$\|$
A section as a ring map of:

$$W_2(R) \xrightarrow{\text{proj}_1} R, \qquad (\mathrm{id}, \delta) : R \dashrightarrow W_2(R)$$

- $\mathbb{Q} \subseteq R$

$$R[t]/t^2 \xrightarrow{\text{mod } t} R, \qquad (\mathrm{id}, d) : R \dashrightarrow R[t]/t^2$$

$\updownarrow 1:1$

Derivations $R \to R$.

- This "section perspective" makes clear that

$$\delta\text{-Ring} \longrightarrow \text{Ring}$$

complete and cocomplete which are the same as the ones computed in Rings.

Lemma δ-Ring $\to$ Set has a left adjoint. (Free δ-ring exists)

Proof Suffices (as Set $= \mathrm{Ind}(*)$) to define value of $F(*)$

$\{ F(*) \longrightarrow A \}$ (another δ-ring)

$\overset{1:1}{\longleftrightarrow}$ (an element of) A — bijection goes by image of x_0

$$\mathbb{Z}\{x\} := \mathbb{Z}[x_0, x_1, x_2, \ldots]$$

$$\phi(x_i) = x_i^p + x_{i+1}$$
$$(\delta(x_i) = x_{i+1})$$ □

Cor $\mathbb{Z}\{x\}$ inherits a (δ) co-ring structure from the one in $\mathbb{Z}[x]$.

Proof

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x_1, x_2]$$
$$x \longmapsto x_1 + x_2$$

$$\mathbb{Z}[x] \longmapsto \mathbb{Z}[x_1, x_2]$$
$$x \longmapsto x_1 \cdot x_2$$

R any Ring, taking (co)-Yoneda

E.g.

$$R \times R \longrightarrow R$$
$$(r_1, r_2) \longmapsto r_1 + r_2$$

Ring $\xrightarrow{F}$ δ-Ring, $\mathbb{Z}[x] \mapsto \mathbb{Z}\{x\}$ is colimit preserving.
$\Rightarrow$ Co-ring struc on $\mathbb{Z}[x]$ induces one $\mathbb{Z}\{x\}$.

$$x_0 \longmapsto x_1 + x_2$$
$$\mathbb{Z}\{x\} \xrightarrow{\hat{+}} \mathbb{Z}\{a, b\}$$
$$x_0 \longmapsto a_0 + b_0$$
$$x_1 \longmapsto \delta(a_0 + b_0) = \delta(\hat{+}(x_0)) = \hat{+}(\delta(x_0))$$ □

Def (Witt vectors)

$$W := \mathrm{Spec}(\mathbb{Z}\{x\}) \text{ ring scheme.}$$
$$\cong \mathbb{A}^{\infty}_{\mathbb{Z}}$$

For any ring R:

$W(R)$ ($= \mathrm{Ring}(\mathbb{Z}\{x\}, R)$) are "the Witt vectors of R".

$$W(R) \xrightarrow[\text{as a set}]{\sim} \prod_{\mathbb{N}} R$$

a) $W(R) \longrightarrow R$, $(x_n) \longmapsto x_0$ ring morphism

Hint to that in exercises:

b) When R is perfect in char p ($R \overset{(-)^p}{\xrightarrow{\sim}} R$)

$W(R)$ is
- p-complete
- p-torsion free
- $W(R)/pW(R) = R$

and $pW(R)$ is

$$0 \to pW(R) \to W(R) \to R \to 0$$

c) $W(R)$ is a δ-Ring

$$(r_n)_{n \in \mathbb{N}} \overset{\delta}{\longmapsto} (r_{n+1})_{n \in \mathbb{N}} \quad \text{(Joyal coordinate)}$$

$\Rightarrow$ ϕ a Frobenius lift on $W(R)$.

Examples $W(\mathbb{F}_p) = \mathbb{Z}_p$

$$W(\mathbb{F}_{q = p^n}) = \mathbb{Z}_p[\zeta_{p^n - 1}]$$

$$W(\overline{\mathbb{F}_p}) = \mathbb{Z}_p^{\text{unram}} = \left(\bigcup_n \mathbb{Z}_p[\zeta_{p^n - 1}]\right)^{\wedge p}$$

Def (Tilt) Let R be a ring

$$(R)^\flat := \lim_{\leftarrow (-)^p} R/pR$$

$$= \{ (\overline{x_n} \in R/pR) \mid \overline{x_{n+1}}^p = \overline{x_n} \}$$

A ring in char p which is perfect.

Remark R is $/\mathbb{F}_p$ and Noetherian domain

Then $R^\flat$ is the smallest perfect subfield of R.

Proposition R p-complete ring ($\cong \lim_{\leftarrow n} R/p^nR$)

$\exists!$ mult map $\#$

$$\begin{tikzcd} R^\flat \arrow[d, dashed, "\#"'] \arrow[dr, "\text{first proj}"] & \\ R \arrow[r, "\text{red. mod } p"'] & R/pR \end{tikzcd}$$

$$\#((\overline{r_n})_{n \in \mathbb{N}}) = \lim_{n \to \infty} (r_n)^{p^n} \in R$$

$$\downarrow \wr$$

$$(\overline{r_n^{p^n}})_{n \in \mathbb{N}} \in \lim_{\leftarrow n} R/p^nR, \quad \overline{r_n^{p^n}} \in R/p^nR$$

Proof Suffice $\forall n$ $\#_n$

$$\begin{tikzcd} R^\flat \arrow[d, dashed, "\#_n"'] \arrow[dr] & \\ R/p^{n+1}R \arrow[r] & R/pR \end{tikzcd}$$

It's constructed for $n-1$, $n \geq 1$

[Key property: A any ring

$a \equiv b \mod I$ for some ideal

Binomial $\Rightarrow$ $a^p \equiv b^p \mod \underline{pI} + I^p$.]

[$I = pA$ $\xrightarrow{\text{by induction}}$ $a \equiv b \mod pA$

$\Rightarrow a^{p^n} \equiv b^{p^n} \mod p^{n+1}A$]

$$\#_n((\overline{r_i})_{i \in \mathbb{N}}) = r_n^{p^n} \quad \text{(any lift)}$$

- Well defined by the above.
- It's coherent with $\#_{n-1}$ by construction.

About unicity: if $\#_n$ and $\#_n'$ are two maps fitting in the diagram, by construction they will be equalized by precomposition by $- \circ (-)^{p^n}$. But $R^\flat$ is perfect — they have to agree. □

Corollary When R is p-complete

$$\lim_{\leftarrow (-)^p} R \xrightarrow{\text{natural}} \lim_{\leftarrow (-)^p} R/pR$$

$$\| $$

$$\{ (x_n \in R) \mid x_{n+1}^p = x_n \}$$

is bijective

$$R^\flat \longrightarrow \lim_{\leftarrow (-)^p} R$$
$$x := (\overline{r_n})_{n \in \mathbb{N}} \longmapsto \left(\#(x^{1/p^n})\right)_{n \in \mathbb{N}}$$

is the inverse. □