Prisms and tilting equivalence

1) Perfectoid rings

Last time: constructed for S perfect

i) $\underbrace{W(S)}_{\overset{\text{as a set.}}{\cong S^{\mathbb{N}}}}$ [Witt vectors of S]

- p-complete, p-torsion free.
- $W(S)/pW(S) \xrightarrow{\sim} S$

ii) $R^\flat = \lim\limits_{\leftarrow (-)^p} R/pR \xrightarrow{\sim} \lim\limits_{\leftarrow (-)^p} R$

with $\#: R^\flat \longrightarrow R$

$(\overline{x_n}) \longmapsto \lim\limits_{n\to\infty} (x_n)^{p^n}$

__Cor__ S perfect.

Teichmüller lift

$\exists!\ [-]: S \longrightarrow W(S)$ mult section of $W(S) \twoheadrightarrow S$ projection.

__Proof__ $(W(S))^\flat = \left(W(S)/pW(S)\right)^\flat = S^\flat = S$

So

$$\begin{tikzcd} S \arrow[d, dashed, "\exists!"'] \arrow[dr, "="] & \\ W(S) \arrow[r] & S \end{tikzcd}$$

from previous prop. $\square$

__Cor__ S perfect. $x \in W(S)$. Then $\exists!\ ([x_i]) \in W(S)^{\mathbb{N}}$

$$x = \sum_{i=0}^{\infty} [x_i] p^i.$$

__Proof__ $W(S) \xrightarrow{\text{mod } p} S$

$x \longmapsto x_0$

$[x_0] \longleftarrow x_0$

So $x - [x_0] = py$ unique by p-tors free.

Continue with $y = y - [y_0]$, and similarly inductively $\square$

__Proposition__ (Fontaine's θ-map)

R p-complete. Then $\exists!$ ring map

$$\begin{tikzcd} W(R^\flat) \arrow[r, "\theta"] & R \\ R^\flat \arrow[u, "{[-]}"] \arrow[ur, "\#"'] & \end{tikzcd}$$

commutes.

__Proof__ Have to be:

$$\theta\left(\sum [x_i] p^i\right) \longmapsto \sum \#(x_i) p^i.$$

Can show "by hand" that it is a ring morphism.

"Simpler"

Deformation theory: As $R^\flat$ perfect

$\forall n$ $W(R^\flat)/p^n W(R^\flat) \longleftarrow \mathbb{Z}/p^n\mathbb{Z}$

"formally étale"

$\Rightarrow \exists!$ lift $R^\flat \longrightarrow R/pR$

as a ring map. Has to be the above $\square$

__Def__ (Distinguished element) S perfect

$d \in W(S)$ is "distinguished" if

$$\delta(d) \in W(S)^\times.$$

__Eg__: $p \in W(S)$; $(x_0, x_1, \dots) \in W(S)$ $x_1 \in S^\times$.

__Def__ (Perfectoid ring) R a ring is "perfectoid" if

i) p-complete

ii) $\exists\, \varpi \in R$ $\varpi^p = pu$ $u \in R^\times$
(and $\varpi \in \operatorname{Im}(R^\flat \to R)$)

iii) $W(R^\flat) \overset{\theta}{\twoheadrightarrow} R$ and
$\ker(\theta) = (d)$ principal

__Remark__ One can show that (ii) implies θ.
d is distinguished. So:
$d = [\varpi^p] + pu$ in the form of the cor. before.

$\to$ Can take ϖ with comp. seq of p-power roots.

__Prop__ (Equiv def) R a ring is pfd if i) and ii) and

iii)' $(-)^p: R/\varpi R \xrightarrow{\sim} R/pR$

$(-)^p: R[\varpi] \xrightarrow{\sim} R[p]$

__Proof__ Omitted in full.

Observe however:

$$\begin{tikzcd} R^\flat/\varpi^\flat \arrow[r, "(A)"] \arrow[d, "(-)^p"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & R/\varpi \arrow[d, "(-)^p"] \\ R^\flat/\varpi^{\flat p} \arrow[r, "(B)"] & R/p \end{tikzcd}$$

If $W(R^\flat)/(d) \to R$ iso then modulo $[\varpi^\flat]$ or $[\varpi^{\flat p}]$ gives (A) and (B). $\square$

__Exercise__ R p-tors free ϖ such that $\varpi^p = pu$.

$R/\varpi \overset{(-)^p}{\hookrightarrow} R/p \iff R \hookrightarrow R[\tfrac{1}{p}]$ p-int closed.

__Examples__ 0) Any perfect ring is.

1) $\mathbb{Z}_p\langle p^{1/p^\infty}\rangle = \left(\mathbb{Z}_p[t^{1/p^\infty}]/(t-p)\right)^{\wedge p}$

So modulo p:

$\mathbb{F}_p[t^{1/p^\infty}]/t$ $\varpi = t^{1/p}$ works.

Tilt: $\mathbb{F}_p\langle t^{1/p^\infty}\rangle$ (t-completed)

2) Closure of int clos domain (exercise).

3) R pfd. $R\langle t^{1/p^\infty}\rangle$

$R/\varpi\langle t^{1/p^\infty}\rangle \overset{(-)^p}{\xrightarrow{\sim}} R/p\langle t^{1/p^\infty}\rangle$ still holds.

4) $\mathbb{Z}_p^{\text{cycl}} = \mathbb{Z}_p\langle \zeta_{p^\infty}\rangle = \mathbb{Z}_p\langle t^{1/p^\infty}\rangle/(t^{p-1} + \dots + 1)$

(See exercises)

2) Prisms, perfect prisms

Def A prism is a pair

(A, I) where A is (p, I)-complete (derived)

where A is a δ-ring and I is a Cartier divisor which is locally generated by a distinguished element. (faithfully flat) $A \to A' \in \delta$-ring $IA' = d$ d, p (rud/$\mathbb{A}$!)

E.g. 1) $(W(S), (p))$ (crystaline)

2) R perfectoid.

Notation $W(R^\flat) := A_{inf}(R)$.

Then $(A_{inf}(R), \ker\theta)$

is a prism.

3) k perfect field.

$(W(k)[[u]], E(u))$
$\phi : u \mapsto u^p$ $\quad$ $\hookrightarrow$ Eisenstein pol for some uniformizer.

E.g. $(\mathbb{Z}_p[[t]], t^n - p)$ $\quad n \in \mathbb{N}$
$t \mapsto t^{1/p}$

Lemma (Rigidity of prisms)

$(A, I) \to (B, J)$ map of prisms.

Then $B \otimes_A I \xrightarrow{\sim} J$ in part $IB = J$.

Proof For simplicity say $I = (d)$

$J = (e)$. (Gen case by faith. flat descent)

So $d = ef$ for some f.

So $\underbrace{\delta(d)}_{unit} = e^p \delta(f) + f^p \underbrace{\delta(e)}_{unit}$
$+ p\,\delta(e)\,\delta(f)$

by (p,e)-completenres (derived)

f^p unit $\Rightarrow$ f unit $\quad \square$

Def A prism is called "perfect"

if ϕ_A is an iso.

Proposition R pfd, $(d) = \ker\theta$.

$$\{\text{Perfect prisms}/A_{inf}(R)\} \xrightarrow[\sim]{\text{mod } d} \{\text{pfd } R\text{-algebras}\}$$

Proof Fix $R \to S$ pfd alg.

Then $A_{inf}(S)/d \to S$ iso

by rigidity of prisms.

Say $A_{inf}(R) \to P$ perfect prism.

$A_{inf}((P/d)^\flat) \to P$

iso modulo p by

$$(P/d)^\flat = \left(P/(p,d)\right)^\flat$$
$$= \left(P/p\right)^{d\wedge} = P/p$$

See exercises. $\quad$ P/p is d-complete as P is (p,d) complete.

$\square$

Corollary R pfd ring.

$\{R^\flat - \varpi^\flat\text{-complete alg}\}$

$\updownarrow \sim \quad \uparrow (-)^\flat$

$\{R \text{ pfd-algebras}\}$

Proof

$\{R^\flat - \varpi^\flat\text{-complete alg}\}$ $\xleftarrow{\text{mod } \theta}$ $\sim$ $\{A_{inf}(R) \text{ perf prisms}\}$

$\sim \uparrow (-)^\flat$ $\quad$ $\swarrow$ mod d

$\{R \text{ pfd-algebras}\}$

$\square$