SHEAF COHOMOLOGY FROM HIGHER SHEAVES

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Abstract. In this short note, we define sheaf cohomology using the higher sheaves perspective (as presented in [HTT]) and deduce from the formalism that Čech cohomology on an acyclic cover computes cohomology. In the first section, we define sheaves on a general ∞-category $\mathcal{C}$ equipped with a Grothendieck topology.

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Foreword. Throughout this note, we denote by Ani the ∞-category of anima, which [HTT] calls *spaces*. Throughout this text we do not make any reference to a peculiar model[1] of higher category theory: this allows the reader to treat ∞-categories much like ordinary categories, and the ∞-category Ani as if it were the category of sets whenever this mental simplification is harmless.

Throughout this note, $\mathcal{C}$ denotes a category equivalent to a *small* ∞-category.

1. Higher sheaves

In this section, we introduce higher sheaves, from [HTT, Section 6.2.2].

1.1. Topologies, Sites and Sheaves of anima.

We use sieves and Grothendieck topologies for categorical convenience and to align with [HTT, Section 6.2.2], which is our main reference on higher sheaves. We also present the more familiar notion of *sites*, which is more customary for algebraic geometers. Namely, readers interested only in sites may focus on Definition 1.9, use Proposition 1.15 as a definition of sheaves, and ignore the rest of Section 1.1.

Definition 1.1 (Sieve)**.** Let $\mathcal{C}$ be an ∞-category. A *sieve* on $\mathcal{C}$ can be equivalently seen ([HTT, Proposition 6.2.2.5]) as follows.

[1] We however cite proven results (in a peculiar model) from [HTT] and [Ker].

(1) A full subcategory $\mathcal{C}^{(0)}$ of $\mathcal{C}$ satisfying that if X is an object of $\mathcal{C}^{(0)}$ and $Y \to X$ is a morphism, then $Y \in \mathcal{C}^{(0)}$
(2) A subobject of $* \in \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$. In other words, this is a functor whose value is either empty or contractible.

The equivalence between the two notions can be seen as follows: given a subobject U of the terminal object $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$, one can define a full subcategory $\mathcal{C}_U$ spanned by all objects $X \in \mathcal{C}$ with $U(X) \neq \emptyset$. A *sieve on* $C \in \mathcal{C}$ is a sieve on $\mathcal{C}_{/C}$.

Remark 1.2. Sieves on $\mathcal{C}$ are in one-to-one correspondence with sieves in $\mathrm{Ho}(\mathcal{C})$. See [Ker, Tag 04UH] and [Ker, Tag 01CP]. A sieve on $C \in \mathcal{C}$ is the same as a sieve on $C \in \mathrm{Ho}(\mathcal{C})$, *i.e.* a sieve on the category $\mathrm{Ho}(\mathcal{C})_{/C}$. This is true even if the natural functor $\mathrm{Ho}(\mathcal{C}_{/C}) \to \mathrm{Ho}(\mathcal{C})_{/C}$ is not an equivalence in general, see [HTT, Remark 6.2.2.3] and [Ker, Tag 04UH].

Example 1.3. Let $\mathcal{C}$ be an ∞-category and $(C_i \to C)_{i \in I}$ be a family of morphisms. Then one can define the *sieve generated by the family of morphisms* $(C_i \to C)_{i \in I}$ as the smallest full subcategory of $\mathcal{C}_{/C}$ containing each $C_i \to C$.

Example 1.4. Let $\mathcal{C} = \mathrm{Op}(X)$ be the poset of open subsets of a topological space X. Let $(U_i \subset X)$ be a collection of opens such that $\bigcup_i U_i = X$. Let's consider the sieve on X generated by $(U_i \subset X)$. This sieve consists of open sets $V \subset X$ such that there exists some index i with $V \subset U_i$. Let's denote the corresponding presheaf by S. In this language we have $S(V) = *$ if and only if there is some i with the property that $V \subset U_i$ and empty otherwise. We jump forward a bit (see Proposition 1.15) to explain to an algebraic geometer why sieves are just a categorical polishing of the usual notion of cover. Namely take any presheaf $\mathcal{F}$ on $\mathrm{Op}(X)$. We claim that

$$\mathrm{Hom}_{\mathrm{PSh}_{\mathrm{Set}}(\mathrm{Op}(X))}(S, \mathcal{F})$$

is a known friend. Namely, one sees that to define a natural transformation $S \to \mathcal{F}$ one has to prescribe some section

$$s_i \colon * = S(U_i) \to \mathcal{F}(U_i)$$

the functoriality will impose that for any $V \subset U_i$ that we have

$$s_V \colon * = S(V) \to \mathcal{F}(V)$$

being equal to $s_{i|V}$. In particular, for $V = U_i \cap U_j$ we see that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$. Therefore, one sees that if $\mathcal{F}$ is a sheaf on X with the usual definition, then the natural map

$$\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{PSh}_{\mathrm{Set}}(\mathrm{Op}(X))}(h_X, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{PSh}_{\mathrm{Set}}(\mathrm{Op}(X))}(S, \mathcal{F})$$

is an isomorphism – and one can define a sheaf if one imposes that this map is an isomorphism for all covering sieves.

Example 1.5. Say $f \colon C \to D$ is any morphism. Let $\mathcal{C}^{(0)}_{/D}$ be a sieve on D. Then we define the *pullback sieve* $f^*\mathcal{C}^{(0)}_{/D}$ as the sieve on C consisting of maps $X \to C$ such that $X \to C \xrightarrow{f} D \in \mathcal{C}^{(0)}_{/D}$.

The following imitates [SGA4-1, Exposé II, Section 1, Définition 1.1].

Definition 1.6 (Grothendieck topology, [HTT, Definition 6.2.2.1])**.** Let $\mathcal{C}$ be an ∞-category. A *Grothendieck topology* or simply a *topology* on $\mathcal{C}$ is the data for each $C \in \mathcal{C}$ of a collection of sieves that are called *covering sieves*. These are required to satisfy the following properties.

(1) If $C \in \mathcal{C}$ then $\mathcal{C}_{/C}$ is a covering sieve of C.
(2) *Local character.* Let $\mathcal{C}^{(0)}_{/C}$ be a covering sieve and $\mathcal{C}^{(1)}_{/C}$ be any sieve on C. If for every $f\colon D \to C$ in $\mathcal{C}^{(0)}_{/C}$ the pullback $f^*\mathcal{C}^{(1)}_{/C}$ is a covering sieve on D, then $\mathcal{C}^{(1)}_{/C}$ is also a covering sieve.
(3) *Stable under base change.* If $f\colon C \to D$ is any morphism and $\mathcal{C}^{(0)}_{/D}$ is a covering sieve, then $f^*\mathcal{C}^{(0)}_{/D}$ is also a covering sieve.

Remark 1.7. Grothendieck topologies on the ∞-category $\mathcal{C}$ are in one-to-one correspondence with Grothendieck topologies on $\mathrm{Ho}(\mathcal{C})$, because of the facts mentioned in Remark 1.2.

Example 1.8. The chaotic topology is the one in which the only covering sieve on C is $\mathcal{C}_{/C}$ for every object $C \in \mathcal{C}$. More interesting examples after Definition 1.9.

Definition 1.9 (Site)**.** Let $\mathcal{C}$ be an ∞-category with fiber products. We say that a *site* or a *pre-topology* on $\mathcal{C}$ is the data for each $C \in \mathcal{C}$ of a set $\mathrm{Cov}(C)$ of families of morphisms $(C_i \to C)$ with target C which are called *coverings* such that

(1) If $D \to C$ is an isomorphism, then $(D \to C) \in \mathrm{Cov}(C)$.
(2) *Local character.* If $(C_i \to C) \in \mathrm{Cov}(C)$ and for each i, we have $(C_{ij} \to C_i) \in \mathrm{Cov}(C_i)$, then $(C_{ij} \to C) \in \mathrm{Cov}(C)$.
(3) *Stable under base change.* If $(C_i \to C) \in \mathrm{Cov}(C)$ and $D \to C$ any morphism, then $(D \times_C C_i \to D) \in \mathrm{Cov}(D)$.

Example 1.10. Let X be a topological space. Let $\mathcal{C} = \mathrm{Op}(X)$. Then for any open U, an open cover $(U_i \to U)$ defines a structure of site.

Example 1.11. Let R be an animated ring. If one does not know what an animated ring is, one may simply consider an ordinary ring. We consider the ∞-category $\acute{\mathrm{E}}\mathrm{t}_R$, the opposite category of the category of étale algebras $R \to S$. Taking as covers finite collection of maps $(R \to S_i)$ such that $R \to \prod_i S_i$ is faithfully flat defines a structure of site on $\acute{\mathrm{E}}\mathrm{t}_R$.

We can consider the smallest Grothendieck topology such that sieves generated by $\mathrm{Cov}(\mathcal{C})$ are covering sieves, as explained in Proposition 1.12. See [SGA4-1, Exposé II, Section 1, Proposition 1.4] for the 1-categorical version and [SAG, A.3.2.1] for an ∞-categorical version, which actually slightly differs from the situation exposed here.

Proposition 1.12. *Let $\mathcal{C}$ be an ∞-category with fiber products equipped with a structure of site. Consider for each object C the collection of sieves S' that contain a sieve S generated by a covering in* $\mathrm{Cov}(\mathcal{C})$*. Then, this collection of sieves defines a Grothendieck topology on $\mathcal{C}$.*

Proof. We want to show properties (1)-(3) of Definition 1.6. Because (1) and (3) from Definition 1.9 hold, we see that (1) and (3) from Definition 1.6 also hold. Therefore we are left to show that property (2) holds as well.

So let $C \in \mathcal{C}$ be an object and $\mathcal{C}^{(0)}_{/C}$ be a covering sieve for the topology defined in the statement and $\mathcal{C}^{(1)}_{/C}$ be any other covering sieve having the property that for any $f\colon D \to C \in \mathcal{C}^{(0)}_{/C}$ the sieve $f^*\mathcal{C}^{(1)}_{/C}$ is a sieve for the topology defined in the statement. By hypothesis, there is some covering $(f_i\colon C_i \to C)$ such that $f_i\colon C_i \to C$ is in $\mathcal{C}^{(0)}_{/C}$. That $f_i^*\mathcal{C}^{(1)}_{/C}$ is a sieve for the topology defined in the statement means that there is some $(C_{ij} \to C_i)$ which is a cover in this sheaf – being in the pullback sheaf means that $C_{ij} \to C$ is a map in $\mathcal{C}^{(1)}_{/C}$. Using this for every i and using (2) of Definition 1.9, we see that $(C_{ij} \to C)$ is a covering, which concludes. □

Now comes the central definition of Section 1.1.

Definition 1.13 (Sheaves of Anima)**.** Let $\mathcal{C}$ be an ∞-category equipped with a Grothendieck topology. The category $\mathrm{Sh}_{\mathrm{Ani}}(\mathcal{C})$ of *sheaves of anima on* $\mathcal{C}$ is defined as the full subcategory of $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani})$ consisting of presheaves $\mathcal{F}$ satisfying the following property: for every object $C \in \mathcal{C}$ and covering sieve $S \to h_C$ of C, the canonical map

$$\mathrm{Map}_{\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})}(h_C, \mathcal{F}) \to \mathrm{Map}_{\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})}(S, \mathcal{F})$$

is an isomorphism in Ani. Another way to phrase this is that if $\mathcal{C}^{(0)}_{/C}$ is the subcategory of $\mathcal{C}_{/C}$ corresponding to S, then the natural map

$$\mathcal{F}(C) \to \varprojlim_{(D\to C)\in \mathcal{C}^{(0)\,\mathrm{op}}_{/C}} \mathcal{F}(D)$$

is an isomorphism.

Remark 1.14. By construction, $\mathrm{Sh}_{\mathrm{Ani}}(\mathcal{C})$ is a localization of $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$ at the morphisms $S \to h_C$ for any covering sieve S. As a localization of a presentable ∞-category, the inclusion $\mathrm{Sh}_{\mathrm{Ani}}(\mathcal{C}) \subset \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$ is a right adjoint. The left adjoint is called the *sheafification*. One can also show that the left adjoint is exact, essentially because (3) in Definition 1.6 ensures that the morphisms we invert are stable under pullback. See [HTT, Lemma 6.2.27] for more details.

When the category $\mathcal{C}$ admits fiber products and the topology comes from a structure of site, we have the more familiar description of sheaves.

Proposition 1.15. *Let $\mathcal{C}$ be an ∞-category with fiber products equipped with a structure of site. Consider the topology on $\mathcal{C}$ associated to the site (see Definition 1.9 and Proposition 1.12). Let $\mathcal{F} \in \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$ be a presheaf. Then the following conditions are equivalent.*

1. *$\mathcal{F}$ is a sheaf.*
2. *For every object $C \in \mathcal{C}$ and every covering $(C_i \to C)$. Consider*
$$\check{\mathrm{C}}_\bullet(C_i \to C) \colon \Delta^{\mathrm{op}} \to \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$$
the Čech nerve of $\coprod_i h_{C_i} \to h_C$ in $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$. Then
$$\mathcal{F}(C) \to \varprojlim_{n\in\Delta} \mathcal{F}(\check{\mathrm{C}}_n(C_i \to C))$$
is an isomorphism in Ani.

Remark 1.16. Let us expand on (2) of Proposition 1.15. Because the category $\mathcal{C}$ has fiber products, let us denote, for $i_1, \ldots, i_{n+1}$ in the indexing set of the covering,

$$C_{i_1\cdots i_{n+1}} := C_{i_1} \times_C \cdots \times_C C_{i_{n+1}}.$$

We have by construction of the Čech nerve that

$$\check{\mathrm{C}}_n(C_i \to C) = \coprod_{i_1,\ldots,i_{n+1}} C_{i_1\ldots i_{n+1}}$$

and therefore that

$$\mathcal{F}(\check{\mathrm{C}}_n(C_i \to C)) = \prod_{i_1,\dots,i_{n+1}} \mathcal{F}(C_{i_1\dots i_{n+1}})$$

To say that $\mathcal{F}(C) \to \varprojlim_{n\in\Delta} \mathcal{F}(\check{\mathrm{C}}_n(C_i \to C))$ is an isomorphism we write

$$\mathcal{F}(C) \longrightarrow \prod_i \mathcal{F}(C_i) \rightrightarrows \prod_{ij} \mathcal{F}(C_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{ijk} \mathcal{F}(C_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} (\cdots)$$

See Example 1.20 for the link with the notion of Set and Groupoid valued sheaves. Note also that by [Ker, Tag 04RE], which states that $\Delta_{\mathrm{inj}} \subset \Delta$ is left cofinal, one may disregard the degeneracies in the above diagram. In other words, in characterization (2) in Proposition 1.15, one can take the limit on Δ_{inj}.

Proof of Proposition 1.15. First, we show the equivalence of (1) with

(2') For every C, and $\mathcal{C}^{(0)}_{/C}$ a sieve generated by a covering of C, we have

$$\mathcal{F}(C) \to \varprojlim_{(D\to C)\in\mathcal{C}^{(0)\,\mathrm{op}}_{/C}} \mathcal{F}(D)$$

The only difference from the sheaf condition is that we check it not for all covering sieves (i.e., sieves containing one generated by a covering) but only for those generated by a covering. Therefore we see that (1) implies (2') trivially.

To show the converse, let $\mathcal{C}^{(1)}_{/C} \supset \mathcal{C}^{(0)}_{/C}$ be a covering sieve containing a sieve generated by a covering $\mathcal{C}^{(0)}_{/C}$. By [Ker, Tag 030Y], it suffices to show that

$$\mathcal{C}^{(1)\,\mathrm{op}}_{/C} \xrightarrow{\mathcal{F}} \mathrm{Ani}$$

is right Kan extended from $\mathcal{C}^{(0)\,\mathrm{op}}_{C/}$. This would means that for every $f\colon D \to C \in \mathcal{C}^{(1)}_{/C}$, we have

$$\mathcal{F}(D) = \varprojlim_{(D'\to D)\in f^*\mathcal{C}^{(0)\,\mathrm{op}}_{/C}} \mathcal{F}(D'). \tag{1}$$

But $f^*\mathcal{C}^{(0)}_{/C}$ is a covering sieve on D because it is generated by $(D \times_C C_i \to D)$ which is a covering by (3) of Definition 1.9 – therefore Equation (1) holds by assumption (2').

Now, the statement follows from [Ker, Tag 04WM], which states that one can more explicitly compute limits of sieves using Čech nerves.

□

1.2. **Valued sheaves.** In this section, we define sheaves with values in an arbitrary ∞-category.

Definition 1.17 (Sheaves with values in a category [SAG, Definition 1.3.1.4])**.** Let $\mathcal{D}$ be a complete ∞-category. The category of $\mathcal{D}$-valued sheaves is the category of limit preserving functors

$$\mathrm{Sh}_{\mathrm{Ani}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{D}.$$

We denote this category by $\mathrm{Sh}_{\mathcal{D}}(\mathcal{C})$. Precomposition with the natural map $\mathcal{C}^{\mathrm{op}} \to \mathrm{Sh}_{\mathrm{Ani}}(\mathcal{C})$[2] induces an equivalence of categories with the full sub category of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ spanned by those

[2]Composition of the Yoneda embedding and the sheafification functor.

functors $\mathcal{F}$ with the properties that for every object $C \in \mathcal{C}$ and covering sieve $\mathcal{C}^{(0)}_{/C}$ the natural map

$$\mathcal{F}(C) \to \varprojlim_{(D \to C) \in \mathcal{C}^{(0)\,\mathrm{op}}_{/C}} \mathcal{F}(D)$$

is an equivalence.

Proof of the last claim in Definition 1.17. Because $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$ is the co-completion of $\mathcal{C}$, colimit preserving functors

$$\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}) \to \mathcal{D}^{\mathrm{op}}$$

corresponds to functors $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$. Thus functors in Definition 1.17 corresponds to functors from $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Now, colimit preserving functors $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}) \to \mathcal{D}^{\mathrm{op}}$ which factor through sheafification are exactly those sending natural maps $S \to h_C$ where S is a covering sieve to an isomorphism. Concatening this with the above translation yields the claim in the definition. $\square$

We have an analogue of Proposition 1.15, with the same proof working.

Proposition 1.18. *Let $\mathcal{C}$ be an ∞-category with fiber products equipped with a structure of site. Consider the topology on $\mathcal{C}$ associated to the site (see Definition 1.9 and Proposition 1.12). Let $\mathcal{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ be a functor. Then the following conditions are equivalent.*

(1) $\mathcal{F}$ is a $\mathcal{D}$-valued sheaf.
(2) For every object $C \in \mathcal{C}$ and every covering $(C_i \to C)$. Consider

$$\check{\mathrm{C}}_\bullet(C_i \to C) \colon \Delta^{\mathrm{op}} \to \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$$

the Čech nerve of $\coprod_i h_{C_i} \to h_C$ in $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$. Then

$$\mathcal{F}(C) \to \varprojlim_{n \in \Delta} \mathcal{F}(\check{\mathrm{C}}_n(C_i \to C))$$

is an isomorphism in $\mathcal{D}$.

Remark 1.19. As in Remark 1.16, it may be more meaningful to write the condition as

$$\mathcal{F}(C) \longrightarrow \textstyle\prod_i \mathcal{F}(C_i) \rightrightarrows \prod_{ij} \mathcal{F}(C_{ij}) \rightrightarrows\!\!\!\to \prod_{ijk} \mathcal{F}(C_{ijk}) \rightrightarrows\!\!\!\rightrightarrows (\cdots)$$

Example 1.20. We take several examples of categories $\mathcal{D}$. This for example shows that the higher sheaf theory recovers the usual theory of sheaves and stacks.

(1) $\mathcal{D} = \mathrm{Set}$. In this case, using Proposition 1.18, $\mathrm{Sh}_{\mathrm{Set}}(\mathcal{C})$ corresponds to presheaves $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, satisfying condition (2). This corresponds therefore to functors $\mathrm{Ho}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ satisfying property (2). But using [Ker, Tag 04RM], we see that this limit actually reduces to the usual sheaf condition

$$\mathcal{F}(C) \longrightarrow \textstyle\prod_i \mathcal{F}(C_i) \rightrightarrows \prod_{ij} \mathcal{F}(C_{ij})$$

meaning that the first arrow is the equalizer of the diagram. Therefore, Set-valued sheaves as defined in Definition 1.17 really correspond to the usual notion of sheaf.

(2) $\mathcal{D} = \mathrm{Ab}$. Because Ab is a 1-category, the above arguments also hold.

(3) $\mathcal{D} = \mathrm{Grpd}$. In this case, the sheaf condition will translate to

$$\mathcal{F}(C) \longrightarrow \prod_i \mathcal{F}(C_i) \rightrightarrows \prod_{ij} \mathcal{F}(C_{ij}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{ijk} \mathcal{F}(C_{ijk})$$

meaning that the first arrow exhibits $\mathcal{F}(\mathcal{C})$ as the limit of the rest of the diagram in the 2-category of groupoids. This is the usual "stack condition". The category $\mathrm{Sh}_{\mathrm{Grpd}}(\mathcal{C})$ is equivalent to the usual category of stacks on $\mathcal{C}$, as in [Sta, Tag 0266].

(4) $\mathcal{D}$ being a n-category. We provide a more conceptual explanation for the behavior showed above. One can show, using a variation on Quillen's theorem A (replace the category which has to be weakly contractible by weakly n-contractible) [Ker, Tag 02NX] that $\Delta^{\leq n} \subset \Delta$ is n-left cofinal. This means that limits with values in n-categories does not change when precomposing with the inclusion $\Delta^{\leq n} \subset \Delta$. Therefore, the sheaf condition for sheaves with values in n-categories can be written as a limit involving only $(n+1)$-"intersections".

(5) $\mathcal{D} = \mathcal{D}(\mathbb{Z})$. In this case, the theory exposed here provides the correct notion of "sheaves of complexes". For example, for a topological space X, with $\mathcal{C} = \mathrm{Op}(X)$ the association of an open to its singular cohomology (seen as a derived object)

$$U \to C^\bullet_{\mathrm{sing}}(U, \mathbb{Z})$$

defines a $\mathcal{D}(\mathbb{Z})$-valued sheaf. The sheaf condition from Proposition 1.18 for a cover by two opens is verified because of the usual Mayer-Vietoris theorem (it's equivalent to it).

2. Sheaf cohomology

Let $\mathcal{C}$ be an ∞-category with a Grothendieck topology and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(\mathcal{C})$ be an abelian sheaf. By the inclusion $\mathrm{Ab} \subset \mathcal{D}(\mathbb{Z})$ one can see $\mathcal{F}$ as an object of $\mathrm{PSh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C})$. Note, however, that $\mathrm{Ab} \subset \mathcal{D}(\mathbb{Z})$ does not preserve limits; therefore, in general if $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(\mathcal{C})$ the composition

$$\mathcal{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab} \to \mathcal{D}(\mathbb{Z})$$

is not a $\mathcal{D}(\mathbb{Z})$-valued sheaf as in Definition 1.17.

We take the following notations.

(1) We denote by

$$c^* \colon \mathrm{PSh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C}) \to \mathrm{Sh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C})$$

the $\mathcal{D}(\mathbb{Z})$-sheafification, *i.e.* the left adjoint to the inclusion

$$\mathrm{Sh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C}) \subset \mathrm{PSh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C}).$$

(2) We denote by

$$p_* \colon \mathrm{Sh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C}) \to \mathcal{D}(\mathbb{Z})$$

the global sections functor, right adjoint to the constant functor $p^* \colon \mathcal{D}(\mathbb{Z}) \to \mathrm{Sh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C})$.

2.1. Sheaf cohomology.

We can now take the following definition of sheaf cohomology of an abelian sheaf.

Definition 2.1 (Sheaf cohomology)**.** Let $C \in \mathcal{C}$ be an object. We define

$$R\Gamma(C, \mathcal{F}) = (c^*\mathcal{F})(C) \in \mathcal{D}(\mathbb{Z}).$$

the *sheaf cohomology of* $\mathcal{F}$ *on* C. The *absolute sheaf cohomology* is defined to be

$$R\Gamma(\mathcal{C}, \mathcal{F}) = p_* c^* \mathcal{F} \in \mathcal{D}(\mathbb{Z}).$$

For $i \in \mathbb{Z}$, we define i-th cohomology groups as

$$\mathrm{H}^i(C,\mathcal{F}) := \pi_{-i}(R\Gamma(C,\mathcal{F})) \quad \mathrm{H}^i(\mathcal{C},\mathcal{F}) := \pi_{-i}(R\Gamma(\mathcal{C},\mathcal{F})).$$

Remark 2.2. The absolute sheaf cohomology is the cohomology at a terminal object of $\mathcal{C}$ if $\mathcal{C}$ has one.

Proposition 2.3 (Coconnectivity)**.** *We have*

$$R\Gamma(C,\mathcal{F}) \in \mathcal{D}_{\leq 0}(\mathbb{Z}),$$

so that the sheaf cohomology of an abelian sheaf is coconnective, i.e. *that* $\mathrm{H}^i(C,\mathcal{F}) = 0$ *if* $i < 0$. *The same holds for* $R\Gamma(\mathcal{C},\mathcal{F})$.

Proof. One can first perform an analogous definition with $\mathcal{D}(\mathbb{Z})_{\leq 0}$. Namely, denote by

$$c^*_{\leq 0}\colon \mathrm{PSh}_{\mathcal{D}(\mathbb{Z})_{\leq 0}}(\mathcal{C}) \to \mathrm{Sh}_{\mathcal{D}(\mathbb{Z})_{\leq 0}}(\mathcal{C})$$

the left adjoint to the inclusion.[3]

The important point is that $\mathcal{D}(\mathbb{Z})_{\leq 0} \subset \mathcal{D}(\mathbb{Z})$ is limit preserving. Therefore for $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(\mathcal{C})$ the composition

$$c^*_{\leq 0}\mathcal{F}\colon \mathrm{Sh}_{\mathrm{Ani}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})_{\leq 0} \to \mathcal{D}(\mathbb{Z})$$

is already limit preserving, *i.e.* a sheaf and therefore insensitive to further sheafification.

The same holds for $R\Gamma(C,\mathcal{F})$ because we always have

$$R\Gamma(\mathcal{C},\mathcal{F}) = \varprojlim_{C\in\mathcal{C}} R\Gamma(C,\mathcal{F}),$$

as we will explain in the proof of Proposition 2.11. Now the claim follows again because $\mathcal{D}_{\leq 0}(\mathbb{Z})$ is stable under limits in $\mathcal{D}(\mathbb{Z})$. $\square$

Definition 2.4 (Acyclic)**.** We say that an abelian sheaf $\mathcal{F}$ is *acyclic* at $C \in \mathcal{C}$ if

$$\pi_i(R\Gamma(C,\mathcal{F})) = 0$$

for $i \neq 0$. Equivalently, the unit map

$$\mathcal{F} \to c_* c^* \mathcal{F}$$

is an isomorphism.

2.2. Čech cohomology.

We now recover one of the most important ways to compute sheaf cohomology.

Definition 2.5 (Čech complex)**.** Let $\mathcal{C}$ be an ∞-category with fiber products equipped with a structure of site. Let $\mathcal{F}$ be an abelian sheaf. Let $(C_i \to C)$ be a covering. We define the Čech complex of $\mathcal{F}$ at the covering $(C_i \to C)$ by

$$C^n_{\check{\mathrm{C}}}((C_i \to C),\mathcal{F}) = \prod_{i_0,\dots,i_n} \mathcal{F}(C_{i_0\cdots i_n})$$

with differential $d\colon C^n_{\check{\mathrm{C}}}((C_i \to C),\mathcal{F}) \to C^{n+1}_{\check{\mathrm{C}}}((C_i \to C),\mathcal{F})$ given by

$$d(s)_{i_0\cdots i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j s_{i_0\cdots\hat{i_j}\cdots i_{n+1}|C_{i_0\cdots i_{n+1}}}.$$

[3]The inclusion $\mathrm{Ab} \subset \mathcal{D}(\mathbb{Z})_{\leq 0}$ is not limit preserving (but colimit preserving), so this adjunction produces a not trivial result.

Denote this complex by $C^\bullet_{\check{C}}((C_i \to C), \mathcal{F})$.

Remark 2.6. A more convenient complex to work with which is quasi-isomorphic to the one from Definition 2.5 is the *ordered Čech complex*, see [Sta, Tag 01FG].

Theorem 2.7 (Čech cohomology)**.** *Let $(C_i \to C)$ be a covering such that for every integer n and indices $i_0, \dots, i_n$ the sheaf $\mathcal{F}$ is acyclic at $C_{i_0 \dots i_n}$. Then, there is canonical isomorphism*

$$R\Gamma(C, \mathcal{F}) \to C^\bullet_{\check{C}}((C_i \to C), \mathcal{F}).$$

Proof. Because $c^*(\mathcal{F})$ is a sheaf, using Proposition 1.18 and Definition 2.1 we get that the natural map

$$R\Gamma(C, \mathcal{F}) \to \varprojlim_{n \in \Delta} R\Gamma(C_{i_0 \cdots i_n}, \mathcal{F}), \tag{2}$$

is an isomorphism. Now, using the acyclicity hypothesis, we get

$$R\Gamma(C, \mathcal{F}) \to \varprojlim_{n \in \Delta} \mathcal{F}(C_{i_0 \cdots i_n}).$$

Now, as this limit in $\mathcal{D}(\mathbb{Z})$ can be realized as the *total complex* associated to the cosimplicial object $\Delta^{\mathrm{op}} \to \mathrm{Ab} \subset \mathrm{Ch}(\mathbb{Z})$ given by $n \mapsto \mathcal{F}(C_{i_0 \cdots i_n})$ and induced restriction maps for the functoriality. A detailed reference on the matter is [Ara25, Section 2]. This total complex as the peculiar feature of being formed for a cosimplicial chain whose elements are discrete chains. Inspecting, one sees that this total complex can be identified to the Čech complex of Definition 2.5. □

Remark 2.8. The formula from Equation (2), together with the fact that limits on Δ are computed by totalization ([Ara25, Section 2]) gives a general recipe to compute cohomology.

Example 2.9. Let M be a manifold and $(U_i \subset M)$ be a *good cover* in the sense that any intersections of the U_i are contractible. Then sheaf cohomology in the constant sheaf $\mathbb{Z}$ (*i.e.* singular cohomology) is computed by the Čech complex

$$\textstyle\prod_i \mathbb{Z} \longrightarrow \prod_{ij} \mathbb{Z} \longrightarrow \prod_{ijk} \mathbb{Z} \longrightarrow (\cdots)$$

Example 2.10. Let X be a separated scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. As the site $\mathrm{Op}(X)$ is equivalent to the site open affines of X and that $\mathcal{F}$ is acyclic at every open affine, we get that the Čech complex on an affine cover computes cohomology, as in [Sta, Tag 01XD].

2.3. Acyclic sites and chaotic topology.

Proposition 2.11 (Chaotic topology comparison)**.** *Let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(\mathcal{C})$ be an abelian group valued sheaf. Suppose that for every object $C \in \mathcal{C}$*

$$\mathrm{H}^i(C, \mathcal{F}) = 0$$

for every $i \in \mathbb{Z}$, meaning that every object of the site is acyclic. Then,

$$R\Gamma(\mathcal{C}, \mathcal{F}) = \varprojlim_{C \in \mathcal{C}} \mathcal{F}(C)$$

in $\mathcal{D}(\mathbb{Z})$. The right hand side is the cohomology with respect to the chaotic topology *on $\mathcal{C}$.*

Proof. Consider the following square of right adjoints commuting up to natural isomorphism.

$$\begin{tikzcd} \mathrm{Sh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C}) \arrow[r, "c_*"] \arrow[dr, "p_*"'] & \mathrm{PSh}_{\mathcal{D}(\mathbb{Z})}(\mathcal{C}) \arrow[d, "\varprojlim"] \\ & \mathcal{D}(\mathbb{Z}) \end{tikzcd}$$

Therefore, it always holds that

$$R\Gamma(\mathcal{C}, \mathcal{F}) = p_* c^* \mathcal{F} = \varprojlim_{C \in \mathcal{C}} R\Gamma(C, \mathcal{F}).$$

But the acyclicity hypothesis implies that the natural map of presheaves $\mathcal{F} \to c_* c^* \mathcal{F}$ is an isomorphism, and therefore

$$R\Gamma(\mathcal{C}, \mathcal{F}) = \varprojlim_{C \in \mathcal{C}} \mathcal{F}(\mathcal{C})$$

□

As an immediate corollary we get the following.

Corollary 2.12. *Let $\mathcal{F}$ be an abelian sheaf for some topology τ on $\mathcal{C}$. Suppose that we are in the situation of Proposition 2.11 and that $\mathcal{F}$ is acyclic at every object of $\mathcal{C}$. Then cohomology of $\mathcal{F}$ taken in any topology which is coarser than τ does not change.*

Example 2.13. Let R be a p-complete ring and AffPfd_R be the opposite of the category of perfectoid R-algebras (cutted of by some strong limit cardinal as in [BM20, Remark 4.18]). Equip AffPfd_R with the p-complete arc-topology [CS23, Section 2.2.1]. Then any perfectoid ring is acyclic [BS22, Proposition 8.10] for the structure sheaf

$$\mathcal{O} \colon S \mapsto S.$$

Therefore the cohomology of R in the arc-topology is equal to

$$\varprojlim_{R \to S} S$$

where $R \to S$ ranges over maps to perfectoid rings and the limit is computed in $\mathcal{D}(\mathbb{Z})$ (or in E_∞-R-algebras). In particular if this limit is discrete, it is an initial perfectoid R-algebra.

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